ENSEMBLE AVERAGED AND MIXTURE THEORY EQUATIONS FOR INCOMPRESSIBLE FLUID–PARTICLE SUSPENSIONS

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Abstract—Two-fluid equations for flowing composites of solid particles in a liquid have been given by mixture theory and by ensemble averaging. The mixture theory equations are postulated and the ensemble averaged equations are derived. The ensemble averaged equations give rise to an explicit form for interaction terms. The equations obtained by the two methods are different. The expression for the stress in a Newtonian fluid with rigid particles is the point at issue. Jackson & Saville, in an appendix to this paper, introduce a method of soft spatial averaging which yields the same result as in the case of ensemble averages.

1. MIXTURE THEORY

Mixture theories are based on an idea of interpenetrating continua in which actual material points are no longer identifiable; the solid and liquid phases are both present at each and every material point. The ideas of mixture theory can be traced back to that branch of mechanics which calls itself rational; an authoritative account is given by Bowen (1971), with more recent developments reviewed in papers by Nunziato *et al.* (1986) and Passman *et al.* (1984). Related ideas based on ensemble averaging can be found in Drew (1983).

Let ρ, V, P and T^* be the true density, velocity, pressure and stress in our flowing composite. Let ϕ be the solids fraction and $\epsilon = 1 - \phi$ the fluids fraction. In mixture theory equations of balance are postulated for interpenetrating fields with variables γ_f, v_f, p_f, T_f, γ_s, v_s, p_s and T_s, where $\gamma_f = \rho_f \epsilon$ and $\gamma_s = \rho_s \phi$ are partial densities. No algorithm is presented for computing interpenetrating fields in the classical approach, say in the approach of Bowen (1971) or Passman *et al.* (1984). Other authors, Anderson & Jackson (1964) and Drew (1983, 1986) notable among them, generate mixture theories from different kinds of averaging, but spatial averaging is most popular. Jackson & Saville have carried out a derivation of local average stresses using a method of soft spatial averaging in the appendix to this paper.

The classical equations of mixture theory for incompressible constituents are:

$$\frac{\partial \gamma_f}{\partial t} + v_f \cdot \nabla \gamma_f + \gamma_f \operatorname{div} v_f = 0 , \tag{1}$$

$$\frac{\partial \gamma_s}{\partial t} + v_s \cdot \nabla \gamma_s + \gamma_s \operatorname{div} v_s = 0 , \tag{2}$$

$$\gamma_f \left(\frac{\partial v_f}{\partial t} + v_f \cdot \nabla v_f \right) = m_f + \gamma_f b_f + \operatorname{div} T_f, \tag{3}$$

$$\gamma_s \left(\frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s \right) = m_s + \gamma_s b_s + \operatorname{div} T_s \tag{4}$$

and

$$\mathbf{m}_f + \mathbf{m}_s = \operatorname{div} \mathbf{S} \,, \tag{5}$$

where $\mathbf{b}_f$ and $\mathbf{b}_s$ are the body force per unit mass (e.g. $\mathbf{b}_f = g$, gravity), $\mathbf{m}$ is the force of interaction between constituents and $\mathbf{S}$ is an interaction stress. The quantities $\mathbf{m}$ and $\mathbf{S}$ are unknown, and are arguably unknowable. They are the basic quantities which need to be modeled in the theory.

We note next that [1] and [2] may be written as

$$\frac{\partial \epsilon}{\partial t} + \operatorname{div} \epsilon \, \mathbf{v}_f = 0 \tag{6}$$

and

$$\frac{\partial \phi}{\partial t} + \operatorname{div} \phi \, \mathbf{v}_s = 0 \,. \tag{7}$$

Hence, by adding them:

$$\operatorname{div} \mathbf{v}_c = 0 \,, \tag{8}$$

where

$$\mathbf{v}_c = \epsilon \, \mathbf{v}_f + \phi \, \mathbf{v}_s \tag{9}$$

is the composite velocity.

We may use [6] and [7] to write [3] and [4] in conservation form:

$$\rho_f \left[\frac{\partial \epsilon \, \mathbf{v}_f}{\partial t} + \operatorname{div}(\epsilon \, \mathbf{v}_f \mathbf{v}_f) \right] = \mathbf{m}_f + \epsilon \rho_f \mathbf{b}_f + \operatorname{div} \mathbf{T}_f \tag{10}$$

and

$$\rho_s \left[\frac{\partial \phi \, \mathbf{v}_s}{\partial t} + \operatorname{div}(\phi \, \mathbf{v}_s \mathbf{v}_s) \right] = \mathbf{m}_s + \phi \rho_s \mathbf{b}_s + \operatorname{div} \mathbf{T}_s \,. \tag{11}$$

We may regard [6], [7], [10], [11] and [5] as the basic system for mixture theories.

2. ENSEMBLE AVERAGED EQUATIONS

Two-fluid equations for incompressible fluid–particle suspensions can be generated by ensemble averaging. The techniques for doing this were set down by Saffman (1971) in his study of flow through porous media and by Lundgren (1972) in his study of dense suspensions and porous media, and in a review article by Drew (1983).

We define an indicator function

$$H(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \text{ is in the solid} \\ 1 & \text{if } \mathbf{x} \text{ is in the fluid} \end{cases} \tag{12}$$

and let $\langle \ \rangle$ designate the operation of taking the average. The average is over many identical trials. We think of an experiment which is started at a certain time. At a later time and at a certain place, we record the value of some flow variable. We repeat the experiment, wait the same time, look at the same place and record again. After many trials we average the values by summing and dividing by the number N of trials, then we let $N \to \infty$. In this manner, we generate a function $\langle \ \rangle (\mathbf{x}, t)$.

Now we get some identities using ensemble averaging and the indicator function, First,

$$\langle H \rangle = \epsilon(\mathbf{x}, t) = 1 - \phi(\mathbf{x}, t)$$

is the fluids fraction and

$$\langle 1 - H \rangle = 1 - \langle H \rangle = \phi(\mathbf{x}, t)$$

is the solids fraction. Recall that $V(x, t)$ is the true velocity. We define an average fluid velocity

$$V_f(x, t) = \frac{\langle HV \rangle}{\langle H \rangle} = \frac{\langle HV \rangle}{\epsilon} \qquad [13]$$

and an average solid velocity

$$V_s(x, t) = \frac{\langle (1 - H)V \rangle}{\langle (1 - H) \rangle} = \frac{\langle (1 - H)V \rangle}{\phi} . \qquad [14]$$

The composite velocity is

$$V_c(x, t) = \langle V \rangle = \langle HV \rangle + \langle (1 - H)V \rangle = \epsilon V_f + \phi V_s. \qquad [15]$$

We may define composite averages and mass averages of any quantity f by

$$f_c = \langle f \rangle = \epsilon f_f + \phi f_s$$

and

$$f_m = \frac{\langle \rho f \rangle}{\langle \rho \rangle} = \frac{(\rho f)_c}{\epsilon \rho_f + \phi \rho_s} .$$

In particular, the mass averaged velocity is

$$V_m = \frac{\langle \rho V \rangle}{\langle \rho \rangle} = \frac{\rho_f V_f \epsilon + \rho_s V_s \phi}{\epsilon \rho_f + \phi \rho_s} . \qquad [16]$$

We next note that $H(x, t)$ is a material variable for materials which do not change phase, always one following fluid particles, always zero following solids; i.e.

$$\frac{\partial H}{\partial t} + V \cdot \nabla H = 0. \qquad [17]$$

Using this, and div $V = 0$, we find

$$0 = \left\langle \frac{\partial H}{\partial t} + V \cdot \nabla H \right\rangle = \left\langle \frac{\partial H}{\partial t} + \text{div } HV \right\rangle$$

$$= \frac{\partial \langle H \rangle}{\partial t} + \text{div } \langle HV \rangle$$

$$= \frac{\partial \epsilon}{\partial t} + \text{div } \epsilon V_f. \qquad [18]$$

In the same way, we may show that

$$\frac{\partial \phi}{\partial t} + \text{div } \phi V_s = 0. \qquad [19]$$

These are the equations of mass balance assumed by mixture theory. It follows that

$$\text{div } V_c = 0. \qquad [20]$$

The reader can prove that

$$\frac{\partial \rho_c}{\partial t} + \text{div } (\rho_c V_m) = 0. \qquad [21]$$

We turn next to the momentum equations. Since

$$\frac{\partial H}{\partial t} + (V \cdot V)H = 0 \quad \text{and} \quad \text{div} V = 0, \qquad [22]$$

we have the identity

$$H \left(\frac{\partial V}{\partial t} + [V \cdot V]V \right) = \frac{\partial HV}{\partial t} + \text{div } HVV . \qquad [23]$$

The momentum equation for the fluid and the solid is

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + [\mathbf{V} \cdot \mathbf{V}]\mathbf{V}\right) = \text{div } T^* + \rho \mathbf{b}. \tag{24}$$

Multiply [24] by H and ensemble average, using [23]:

$$\rho_f \frac{\partial}{\partial t} \langle H\mathbf{V}\rangle + \rho_f \text{div}\langle H\mathbf{V}\mathbf{V}\rangle = \langle H \text{ div } \mathbf{T}^*\rangle + \rho_f \mathbf{b}_f \epsilon. \tag{25}$$

Now we differentiate by parts

$$\langle H \text{ div } \mathbf{T}^*\rangle = \text{div}\langle H\mathbf{T}^*\rangle - \langle \mathbf{V}H \cdot \mathbf{T}^*\rangle \tag{26}$$

where

$$\mathbf{V}H = \delta_\Sigma(\mathbf{x})\mathbf{n}, \quad \mathbf{V}\langle H\rangle = \langle \mathbf{V}H\rangle = \langle \delta_\Sigma \mathbf{n}\rangle = \mathbf{V}\epsilon(\mathbf{x}, t), \tag{27}$$

$\delta_\Sigma(\mathbf{x})$ is a one-dimensional Dirac delta-function across the solid–fluid interface and $\mathbf{n}$ is the outward normal to the solid. We next note that $\mathbf{n} \cdot \mathbf{T}^* = \mathbf{t}$ is the traction vector at a point $\mathbf{x}_\Sigma$ on the interface. From the definitions of $\mathbf{T}_f^*$ we have

$$\langle H\mathbf{T}^*\rangle = \epsilon \mathbf{T}_f^*(\mathbf{x}, t). \tag{28}$$

Using these relations, we may write [25] as

$$\rho_f\left(\frac{\partial}{\partial t}\epsilon \mathbf{V}_f + \text{div}\langle H\mathbf{V}\mathbf{V}\rangle\right) = \text{div } \epsilon \mathbf{T}_f^* - \langle \delta_\Sigma(\mathbf{x})\mathbf{t}\rangle + \rho_f \epsilon \mathbf{b}_f. \tag{29}$$

Using the same method, we find a momentum equation for the solid in the form

$$\rho_s\left(\frac{\partial \phi \mathbf{V}_s}{\partial t} + \text{div}\langle(1 - H)\mathbf{V}\mathbf{V}\rangle\right) = \text{div } \phi \mathbf{T}_s^* + \langle \delta_\Sigma(\mathbf{x})\mathbf{t}\rangle + \rho_s \phi \mathbf{b}_s. \tag{30}$$

3. COMPARISON OF THE ENSEMBLE AVERAGED AND MIXTURE THEORY EQUATIONS

If we suppose that the equations of these two theories are the same, we may conclude that

$$\epsilon \mathbf{T}_f^* = \mathbf{T}_f, \quad \mathbf{V}_f = \mathbf{v}_f,$$

$$\phi \mathbf{T}_s^* = \mathbf{T}_s, \quad \mathbf{V}_s = \mathbf{v}_s,$$

$$\mathbf{m}_f = \rho_f \text{div}[\epsilon \mathbf{v}_f \mathbf{v}_f - \langle H\mathbf{V}\mathbf{V}\rangle] - \langle \delta_\Sigma \mathbf{t}\rangle, \tag{31}$$

$$\mathbf{m}_s = \rho_s \text{div}[\phi \mathbf{v}_s \mathbf{v}_s - \langle(1 - H)\mathbf{V}\mathbf{V}\rangle] + \langle \delta_\Sigma \mathbf{t}\rangle \tag{32}$$

and

$$\text{div}\{\mathbf{S} - \rho_f \epsilon \mathbf{v}_f \mathbf{v}_f - \rho_s \phi \mathbf{v}_s \mathbf{v}_s + \langle[\rho_f H + \rho_s(1 - H)]\mathbf{V}\mathbf{V}\rangle\} = 0. \tag{33}$$

Equations [31]–[33] are previously unknown relations between the interaction terms and the "Reynolds stresses". The interaction stress $\mathbf{S}$ in [33] cannot be taken to be zero because of the fact that the ensemble average of a dyadic product is not the product of ensemble averages. At this stage, there is no contradiction between the equations of mixture theory and the ensembled averaged equations.

Let us assume now that the fluid phase is Newtonian,

$$\mathbf{T^*} = -P\mathbf{1} + 2\mu\mathbf{D[V]} \tag{34}$$

in the fluid; and the solid phase is a rigid body, for which

$$\mathbf{D[V]} = 0 \tag{35}$$

on solids; where

$$\mathbf{D[V]} = \tfrac{1}{2}(\nabla\mathbf{V} + [\nabla\mathbf{V}]^\mathsf{T}) \tag{36}$$

is the rate of strain.

The stress for the fluid phase in mixture theory is given by Nunziato *et al.* (1986):

$$\mathbf{T_f} = -\epsilon p_f\mathbf{1} + 2\epsilon\mu\mathbf{D[v_f]}. \tag{37}$$

This differs from the fluid stress arising from ensemble averaging:

$$\mathbf{T_f} = \epsilon\mathbf{T_f^*} = \langle H\mathbf{T^*}\rangle = \langle H(-P\mathbf{1} + 2\mu\mathbf{D[V]})\rangle$$

$$= -\epsilon P_f\mathbf{1} + 2\mu\langle(H-1)\mathbf{D[V]}\rangle + 2\mu\langle\mathbf{D[V]}\rangle$$

$$= -\epsilon P_f\mathbf{1} + 2\mu\mathbf{D[\langle V\rangle]}$$

$$= -\epsilon p_f\mathbf{1} + 2\mu\mathbf{D[v_c]} \tag{38}$$

where we have put $P_f = p_f$, $\langle\mathbf{V}\rangle = \mathbf{V_c}$ and

$$\langle(H-1)\mathbf{D[V]}\rangle = 0 \tag{39}$$

because $H - 1$ is zero in the fluid and $D[\mathbf{V}] = 0$ in the solid. The step $\langle\mathbf{D[V]}\rangle = \mathbf{D[\langle V\rangle]}$ is true because $\mathbf{V}$ is continuous, $D[\mathbf{V}]$ is uniformly bounded.

The stresses given by [37] and [38] are generally different because

$$\mathbf{D[v_c]} = \mathbf{D[\epsilon v_f + \phi v_s]} \tag{40}$$

is different from $\epsilon\mathbf{D}[\mathbf{v_f}]$.

Another expression for the stress in the fluid phase of a Newtonian fluid, based on volume averaging (Ishii 1975), has been given by Drew (1983, 1986):

$$\mathbf{T_f} = -\epsilon p_f\mathbf{1} + \mu\{2\mathbf{D[v_f]} + (\nabla\phi)[\mathbf{v_f} - \mathbf{v_s}] + [\mathbf{v_f} - \mathbf{v_s}](\nabla\phi)\}. \tag{41}$$

This expression also differs from the one, [38], which arises from ensemble averaging. Yet another expression for the stress in the fluid phase which agrees with that via ensemble averaging is derived from spatial averaging by Jackson & Saville in the appendix to this paper.

In the modeling of multiphase flows of rigid particles, it is convenient to write

$$\mathbf{T^*} = -P\mathbf{1} + \tau, \tag{42}$$

where P is the mean normal stress. The ensemble average of this is

$$\mathbf{T_s} = \phi\mathbf{T_s^*} = -\phi p_s\mathbf{1} + \langle(1-H)\tau\rangle \tag{43}$$

where we have assumed that $P_s = p_s$ and

$$\phi P_s = \langle(1-H)P\rangle. \tag{44}$$

Expression [43] with $\tau = 0$ is frequently postulated in mixture theories (e.g. Nunziato *et al.* 1986; Passman *et al.* 1984). Givler (1987) argues that p_s may be interpreted as the average of the local pressure field around an isolated particle. Perhaps it is justified to think of the components of the deviatoric stress as a perturbation of p_s which scales with the particle size.

If we assume [38], [39] and [43], and manipulate the inertia terms to a more elegant form, we get the following system of ensemble averaged equations:

$$\frac{\partial \epsilon}{\partial t} + \text{div } \epsilon \mathbf{v}_f = 0,$$ [45]

$$\frac{\partial \phi}{\partial t} + \text{div } \phi \mathbf{v}_s = 0,$$ [46]

$$\rho_f \epsilon \left(\frac{\partial \mathbf{v}_f}{\partial t} + \mathbf{v}_f \cdot \nabla \mathbf{v}_f \right) + \rho_f \text{div}\langle H(\mathbf{V} - \mathbf{v}_f)(\mathbf{V} - \mathbf{v}_f) \rangle = -\nabla(\epsilon p_f) + \mu \nabla^2 \mathbf{v}_c - \langle \delta_\Sigma(\mathbf{x})\mathbf{t} \rangle + \epsilon \rho_f \mathbf{b}_f$$ [47]

and

$$\rho_s \phi \left(\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \right) + \rho_s \text{div}\langle (1 - H)(\mathbf{V} - \mathbf{v}_s)(\mathbf{V} - \mathbf{v}_s) \rangle$$

$$= -\nabla(\phi p_s) + \langle \delta_\Sigma(\mathbf{x})\mathbf{t} \rangle + \text{div}\langle (1 - H)\tau \rangle + \phi \rho_s \mathbf{b}_s.$$ [48]

The addition of a constant pressure to the system as a whole, to p_f, p_s and $\mathbf{t}$ simultaneously, has no dynamic consequence. The proof of this uses [27].

When we add [45] and [46] we get

$$\text{div } \mathbf{v}_c = 0.$$ [49]

Equations [47]–[49] can be regarded as effective for the composite media. The boundary condition between the fluid and the particle takes form in the traction vector term in [47] and [48] and it is probably best not to combine the two equations.

The existence of two fluid equations even when one of the fluids is solid is perfectly justified by ensemble averaging. These equations, like other two-fluid models, are not closed and methods of closure, or constitutive models for the interaction terms, are required to put the equations into a form suitable for applications. Moreover, since averaging over repeated identical trials is not a realizable proposition, the ensemble averaged variables are conceptually abstract and their relation to more physically intuitive variables, like the ones which arise from spatial averaging, is uncertain.

There are many problems of mechanics in which averaging of any kind is inappropriate, leading to correct but irrelevant statements, like "average weather is widely scattered showers" or "the average gender is slighly female".

Finally, we shall list a summary of the results presented in this paper. Relation [27] is new and leads to the proof (under [48]) that the addition of a constant pressure to the system does not change the ensemble averaged equations. The relations, [31]–[33], between the interaction force and stress and the Reynolds stresses which arise in ensemble averaging are new. The demonstration, [38], that the viscous part of the ensembled averaged stress in a Newtonian fluid with rigid particles is determined by the fluid viscosity times the symmetric part of the gradient of the composite velocity is new and different from equivalent expressions which arise from mixture theory and volume averaging. The expressions for the Reynolds stresses in terms of dyadic products of the differences between the fluid velocity and various average velocities are new, as are the equations for the composite material.

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APPENDIX

Local Average Stresses in an Interdispersion of Two Newtonian Fluids

We consider a dispersion of drops of a fluid of viscosity μ_d in a second fluid of viscosity μ_c. If a denotes some physical property, measurable at each point of either fluid, then intrinsic local average values of this quantity for each phase, denoted by $\{a\}_d$ and $\{a\}_c$, are defined by

$$\phi(\mathbf{x}, t)\{a\}_d(\mathbf{x}, t) = \int_{V_d} g(|\mathbf{x} - \mathbf{y}|)a(\mathbf{y}, t)\,\mathrm{d}V_y$$

and

$$\epsilon(\mathbf{x}, t)\{a\}_c(\mathbf{x}, t) = \int_{V_c} g(|\mathbf{x} - \mathbf{y}|)a(\mathbf{y}, t)\,\mathrm{d}V_y\,.$$

Here ϕ and ϵ denote the volume fractions of the respective phases, V_d and V_c are the regions of space occupied by them and $g(r)$ is a weighting function which defines the size of the averaging region. The sum $V = V_d + V_c$ represents the entire region occupied by the system of interest, and the boundary of V will be denoted by S.

The weighting function $g(r)$ is positive, monotone decreasing and tends to zero as r tends to infinity. It may be required to be differentiable up to any desired order, and each derivative is integrable over the semi-infinite interval of positive r. It is also normalized so that

$$4\pi \int_0^\infty g(r)r^2\,\mathrm{d}r = 1$$

and its *radius*, r_0, is defined by

$$4\pi \int_0^{r_0} g(r)r^2\,\mathrm{d}r = 4\pi \int_{r_0}^\infty g(r)r^2\,\mathrm{d}r = 1/2.$$

A characteristic dimension L of the region V defines the scale of the system as a whole, and we assume that there is a second scale l, large compared with the spacing of the drops of the dispersed phase but very small compared with L, such that the values of local average quantities are essentially independent of r_0 and the particular algebraic form of $g(r)$, provided $l \ll r_0 \ll L$ and the distance of x from the nearest point of S is much larger than r_0. If these conditions are not satisfied, values of the local average variables, even at points remote from the system boundary, are sensitive to detalis of the weighting function, and the procedure is inappropriate. $g(r)$ will usually be chosen to fall to zero quite rapidly when r exceeds r_0, and the limiting case of a Heaviside function of $r_0 - r$ gives the usual average over a hard sphere.

We consider the case in which the property a is a component of the fluid velocity $\mathbf{u}$, when

$$\frac{\partial}{\partial x_j}[\phi\{u_i\}_d(\mathbf{x}, t)] = \int_{V_d} \frac{\partial}{\partial x_j} g(|\mathbf{x} - \mathbf{y}|) u_i(\mathbf{y}, t) \, dV_y = -\int_{V_d} u_i(\mathbf{y}, t) \frac{\partial}{\partial y_j} g(|\mathbf{x} - \mathbf{y}|) \, dV_y$$

$$= -\int_{V_d} \frac{\partial}{\partial y_j}[g(|\mathbf{x} - \mathbf{y}|) u_i(\mathbf{y}, t)] \, dV_y + \int_{V_d} g(|\mathbf{x} - \mathbf{y}|) \frac{\partial}{\partial y_j} u_i(\mathbf{y}, t) \, dV_y.$$

There is an obvious similar expression for $\partial/\partial x_j[\epsilon\{u_i\}_c)(\mathbf{x}, t)]$ and, combining this with the above and with the corresponding two expressions with the suffixes i and i interchanged, we find

$$\frac{\partial}{\partial x_i}[\phi\{u_i\}_d] + \frac{\partial}{\partial x_i}[\phi\{u_j\}_d] + \frac{\partial}{\partial x_j}[\epsilon\{u_i\}_c] + \frac{\partial}{\partial x_i}[\epsilon\{u_j\}_c]$$

$$= \int_{V_d} g\left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i}\right) dV_y + \int_{V_c} g\left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i}\right) dV_y - \int_V \left[\frac{\partial}{\partial y_j}(gu_i) + \frac{\partial}{\partial y_i}(gu_j)\right] dV_y.$$

The last term in this case can be written as a surface integral

$$\int_s g(u_i n_j + u_j n_i) \, dS,$$

where $\mathbf{n}$ is the unit outward normal to the system at its boundary. This integral is clearly negligibly small when the shortest distance of $\mathbf{x}$ from the surface S is much larger than r_0. Then, since $\partial u_i/\partial y_j + \partial u_j/\partial y_i$ is equal to E_{ij}/μ_d in the dispersed phase and E_{ij}/μ_c in the continuous phase, where $\mathbf{E}$ is the stress tensor, the above relation can be written as

$$\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} = \frac{\phi}{\mu_d}\{E_{ij}\}_d + \frac{\epsilon}{\mu_c}\{E_{ij}\}_c,$$

where $w = \phi\{\mathbf{u}\}_d + \epsilon\{\mathbf{u}\}_c$.

This is the final result. The case of solid particles dispersed in a Newtonian fluid is approached in the limit as $\mu_d \to \infty$, and the result is seen to be the same as in the case of ensemble averages.